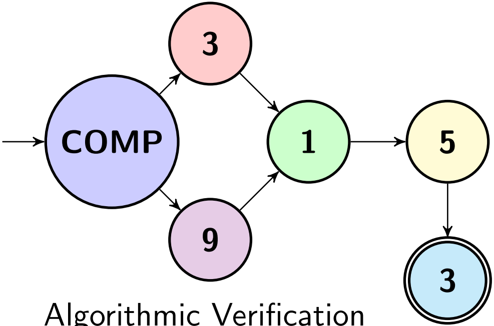

Algorithmic Verification

Timed Automata and Languages

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Term 1 2020

Timed Systems

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For *dense time* systems, we care about a real-valued *continuous* time clocks.

Example (Dense Time System)

A light controlled by one button, where a “double press” of the button increases the brightness of the light. The second button press must be at most 3 time units after the first button press for the “double press” behaviour to trigger.
After 12 time units, the light must turn off.

Can we get away with discrete time?

No

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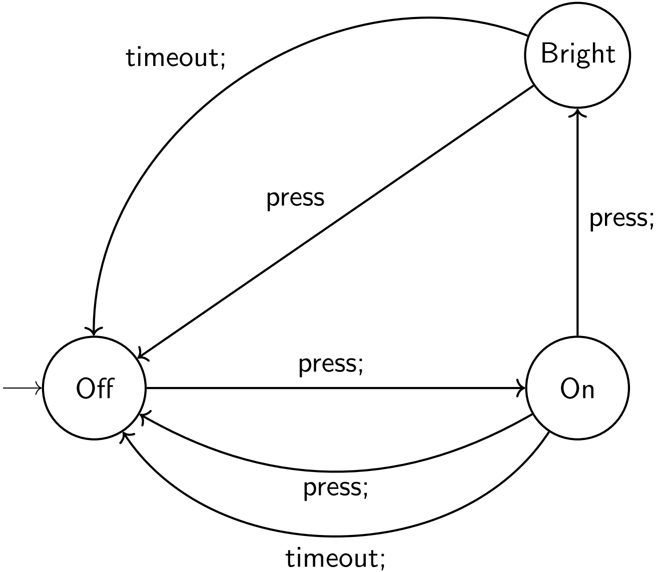
No

Theorem (Brzozowski and Seger)

For every $k \geq 1$ there is a system where the set of states reachable in dense time is strictly larger than the set of states reachable in discrete time in $\frac{1}{k}$ steps.
This is shown for asynchronous circuits, but applies generally.

Towards Timed Automata



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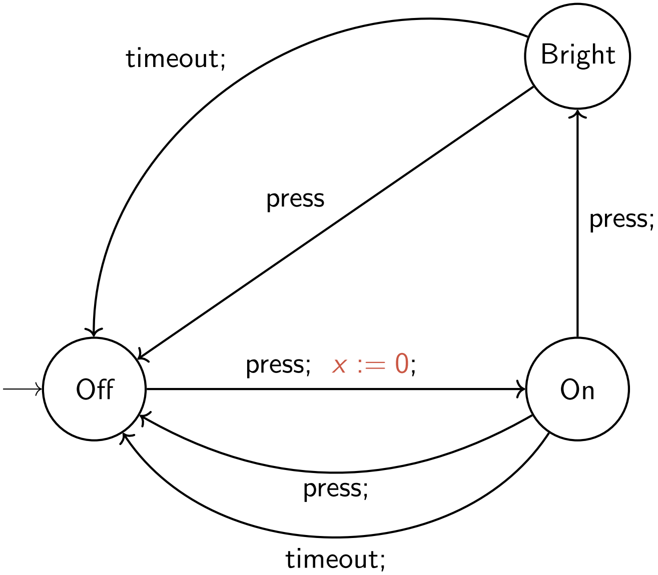

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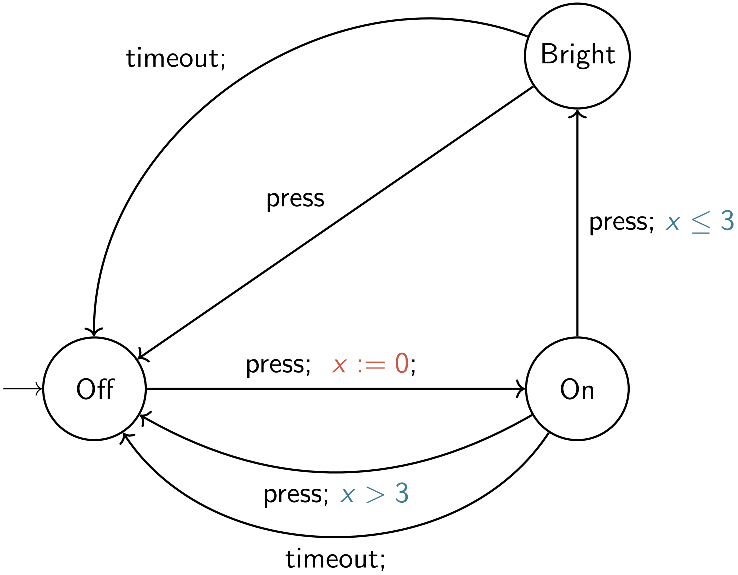

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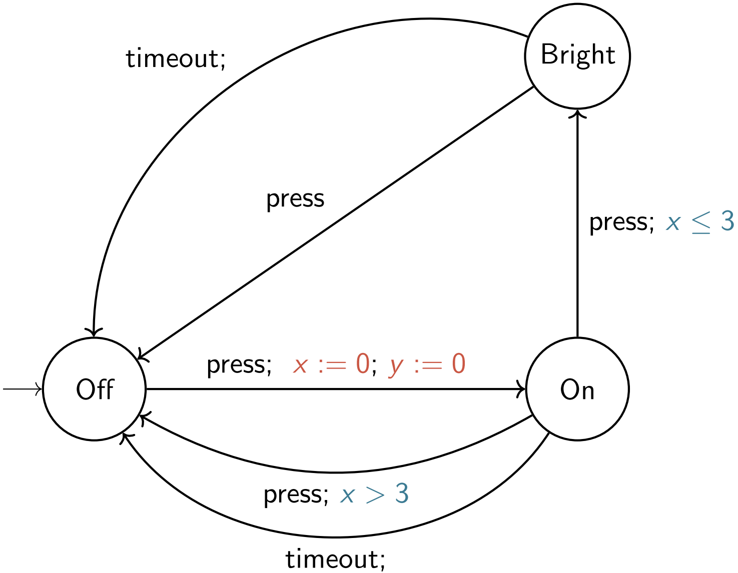

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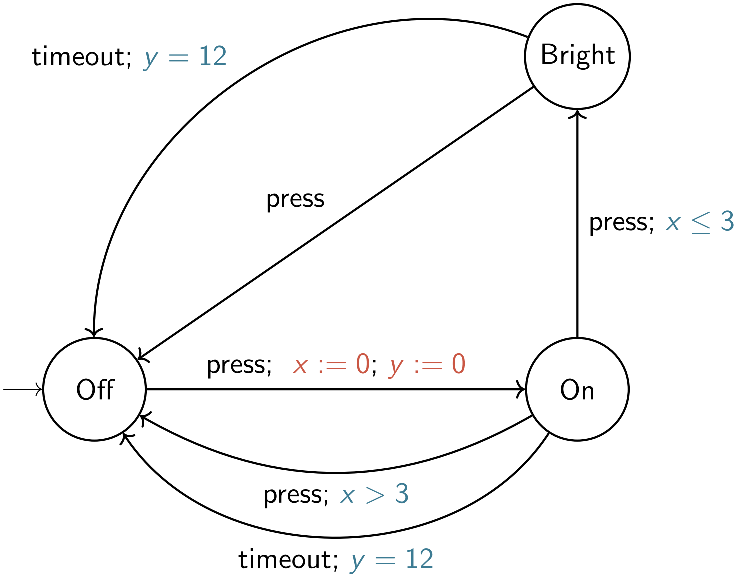

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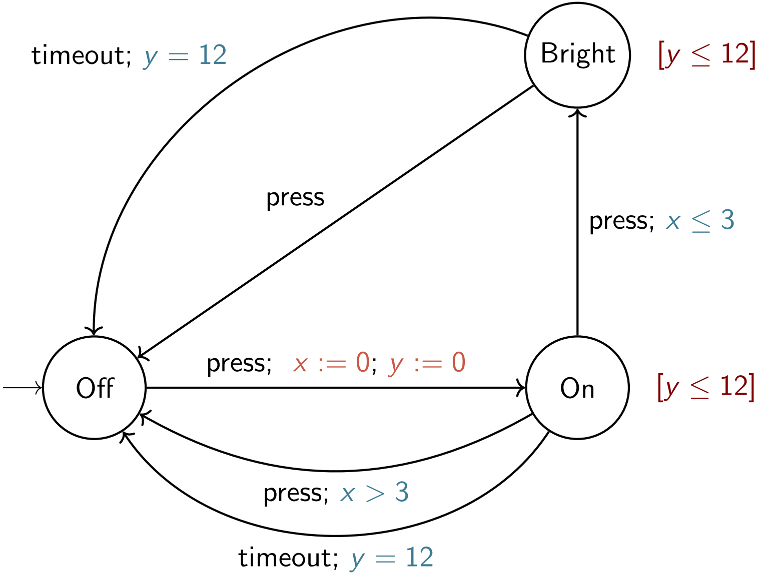

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Definition

A timed automaton $\mathcal{A}$ is a 6-tuple $(L, \ell_0, \mathrm{Act}, X, \mathrm{Inv}, \longrightarrow)$ where:

- L is a set of locations.
- ℓ_0 is the initial location.
- Act is the set of discrete actions.
- X is the set of clock variables.
- $\mathrm{Inv}(\ell)$ is a *clock constraint* invariant associated with ℓ.
- Transitions are defined as $\ell \xrightarrow{g;a;r} \ell'$ where
  - g is zero or more clock constraint *guards*.
  - a is an action $\in$ Act
  - r is zero or more *clock resets*

Clock Constraints

For reasons that will become clear later, we want to restrict clock constraints to linear subtractions:

$$\varphi ::= x \sim k \mid x - y \sim k \mid \varphi_1 \wedge \varphi_2$$

where $x, y \in X$ and $k \in \mathbb{Z}$ and $(\sim) \in \{<, \leq, =, \geq, >\}$

States and Runs

States

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$$(\text{Off}, x = y = 3.2) \xrightarrow{27.87} (\text{Off}, x = y = 31.07) \longrightarrow \cdots$$

Product

The product of timed automata is as the discrete time product, where we take the conjunction of all guards and invariants, and the union of all resets.

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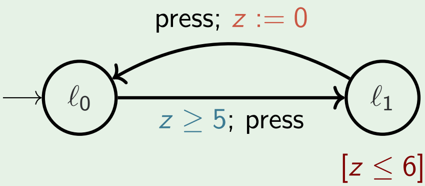
Example (Boardwork)

Let's compute the product of the light automaton with this user automaton:



Timed Words and Languages

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A timed word is a finite or infinite sequence of pairs of actions and time stamps.

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Example (overlines indicate sequences)

$$L_1 = \{\overline{(a,t)} \mid \overline{a} = (01)^*0 \land \forall i \geq 0.\ t_{2(i+1)} - t_{2i} = 3\}$$
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Timed Languages

Extend TA definition of $\mathcal{A}$ with a set F of final states and a set R of repeating states.

- A finite word w is $\in \mathcal{L}(\mathcal{A})$ iff a run generating the word w ends in a state F.
- An infinite word w is $\in \mathcal{L}(\mathcal{A})$ iff a run generating the word w visits states in R infinitely often.

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Problem

Write a timed automaton for words on the alphabet $\{a, b\}$ that contain two a's exactly 1 time unit apart.
Complement is not timed regular $\Rightarrow$ not closed.

ε-Transitions

Unlike for discrete time systems, ε transitions add power for timed automata.

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Example

Consider the language where actions must occur on integer time stamps. This can be done with a ε reset, but cannot be expressed as a timed automaton without ε.

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This is because our timed words only pair time stamps with discrete actions, so violating invariants by sitting still does not change the set of recognised words.
So, we just move the invariants to both the incoming and outgoing transitions like so:

$$\ell_1 \xrightarrow{g;a;r} \ell_2$$

becomes

$$\ell_1 \xrightarrow{g \ \land \ \mathsf{Inv}(\ell_1) \ \land \ r(\mathsf{Inv}(\ell_2));a;r} \ell_2$$

Where $r(\varphi)$ is applying the resets r as a substitution to φ.